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Convergent and divergent numbers games for certain collections of edge-weighted graphs

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Abstract

The numbers game is a one-player game played on a finite simple graph with certain "amplitudes" assigned to its edges and with an initial assignment of real numbers to its nodes. The moves of the game successively transform the numbers at the nodes using the amplitudes in a certain way. Here, the edge amplitudes will be negative integers. Combinatorial methods are used to investigate the convergence and divergence of numbers games played on certain such graphs. The results obtained here provide support for results in a companion paper.

Keywords: numbers game, generalized Cartan matrix, Dynkin diagram

Contents

1. Introduction, definitions, and preliminary results

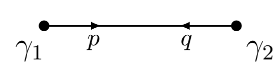
The numbers game is a one-player game played on a finite simple graph with weights (which we call "amplitudes") on its edges and with an initial assignment of real numbers to its nodes. Here, each of the two edge amplitudes (one for each direction) will be negative integers. The move a player can make is to "fire" one of the nodes with a positive number. This move transforms the number at the fired node by changing its sign, and it also transforms the number at each adjacent node in a certain way using an amplitude along the incident edge. The player fires the nodes in some sequence of the player's choosing, continuing until no node has a positive number.

The numbers game as formulated by Mozes [Moz] has also been studied by Proctor [Pro1], [Pro2], Björner [Björ], Eriksson [Erik1], [Erik2], [Erik3], [Erik4], [Erik5], [Erik6], [DE], Wildberger [Wil1], [Wil2], [Wil3], and Donnelly [Don2]. Wildberger studies a dual version which he calls the "mutation game." See Alon *et al* [AKP] for a brief and readable treatment of the numbers game on "unweighted" cyclic graphs. Much of the numbers game discussion in §4.3 of the book [BB] by Björner and Brenti can be found in [Erik2] and [Erik5]. See these references for discussions of how the numbers game is a combinatorial encoding of information for geometric representations of Weyl groups (and more generally Coxeter groups) and has uses for computing orbits, finding reduced decompositions of Weyl group elements, solving the word problem, and obtaining combinatorial models for Weyl groups. Proctor developed this process in [Pro1] to compute Weyl group orbits of weights with respect to the fundamental weight basis. Here we use his perspective of firing nodes with positive, as opposed to negative, numbers. Mozes studied numbers games on graphs for which the matrix M of integer amplitudes is "symmetrizable" (i.e. there is a nonsingular diagonal matrix D such that $D^{-1}M$ is symmetric); in [Moz] he obtained "strong convergence" results and a geometric characterization of the initial positions for which the game terminates. There will be no symmetrizable assumption here.

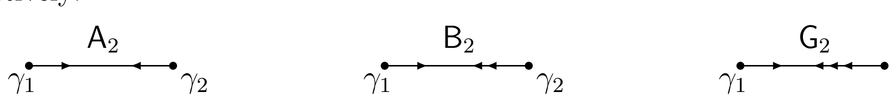
Such graphs-with-amplitudes will henceforth be called "GCM graphs" for reasons explained below. Given any such graph, an initial "position" is an assignment of numbers to the nodes. The position is is "nonzero" if at least one of the numbers is nonzero. A numbers game played from some initial position is "convergent" if it terminates after a finite number of node firings; otherwise we say the game is "divergent."

Here we investigate convergence and divergence of numbers games played on certain GCM graphs. The purpose is to provide supporting details for the proof of a result of [DE]. In particular, we aim to give straightforward combinatorial proofs of Propositions 2.3 and 3.1. These results are used in the proof of the first main result of [DE]: A connected GCM graph has a convergent numbers game played from a nonzero initial position with nonnegative numbers if and only if the graph is one of the "Dynkin diagrams" of Figure 1.1, in which case all numbers games played from a given initial position will converge to the same terminal position in the same number of steps. Proposition 2.3 asserts that for the GCM graphs in Figure 1.1, all numbers games are convergent. Applying results of Eriksson, we will then see that two numbers games played from the same initial position on one of these graphs converge to the same terminal position in the same number of steps. Proposition 3.1 asserts that for the GCM graphs of Figure 3.1, a numbers game is divergent from any nonzero initial position with nonnegative numbers. Two key results needed for proofs of both propositions are Eriksson's Strong Convergence and Comparison Theorems (see Theorems 1.1 and 1.3 below). Proofs of both propositions also involve case analysis arguments that are fairly routine, can be checked by hand, and are often easily expedited using a computer algebra system to automate some of the computations. Complete details are provided here.

Fix a positive integer n and a totally ordered set I_n with n elements (usually $I_n := \{1 < \ldots < n\}$). A *generalized Cartan matrix* (or *GCM*) is an $n \times n$ matrix $M = (M_{ij})_{i,j \in I_n}$ with integer entries satisfying the requirements that each main diagonal matrix entry is 2, that all other matrix entries are nonpositive, and that if a matrix entry M_{ij} is nonzero then its transpose entry M_{ji} is also nonzero. Generalized Cartan matrices are the starting point for the study of Kac–Moody algebras: beginning with a GCM, one can write down a list of the defining relations for a Kac–Moody algebra as well as the associated Weyl group (see [Kac] or [Kum]). To an $n \times n$ generalized Cartan matrix $M = (M_{ij})_{i,j \in I_n}$ we associate a finite graph Γ (which has undirected edges, no loops, and no multiple edges) as follows: The nodes $(\gamma_i)_{i \in I_n}$ of Γ are indexed by the set I_n, and an edge is placed between nodes γ_i and γ_j if and only if $i \neq j$ and the matrix entries M_{ij} and M_{ji} are nonzero. We call the pair (Γ, M) a *GCM graph*. We consider two GCM graphs $(\Gamma, M = (M_{ij})_{i,j \in I_n})$ and $(\Gamma', M' = (M'_{pq})_{p,q \in I'_n})$ to be the same if under some bijection $\sigma : I_n \to I'_n$ we have nodes γ_i and γ_j in Γ adjacent if and only if $\gamma'_{\sigma(i)}$ and $\gamma'_{\sigma(j)}$ are adjacent in Γ' with $M_{ij} = M'_{\sigma(i),\sigma(j)}$. With $p = -M_{12}$ and $q = -M_{21}$, we depict a generic connected two-node GCM graph as follows:



We use special names and notation to refer to two-node GCM graphs which have $p = 1$ and $q = 1$, 2, or 3 respectively:



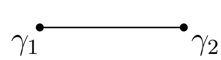
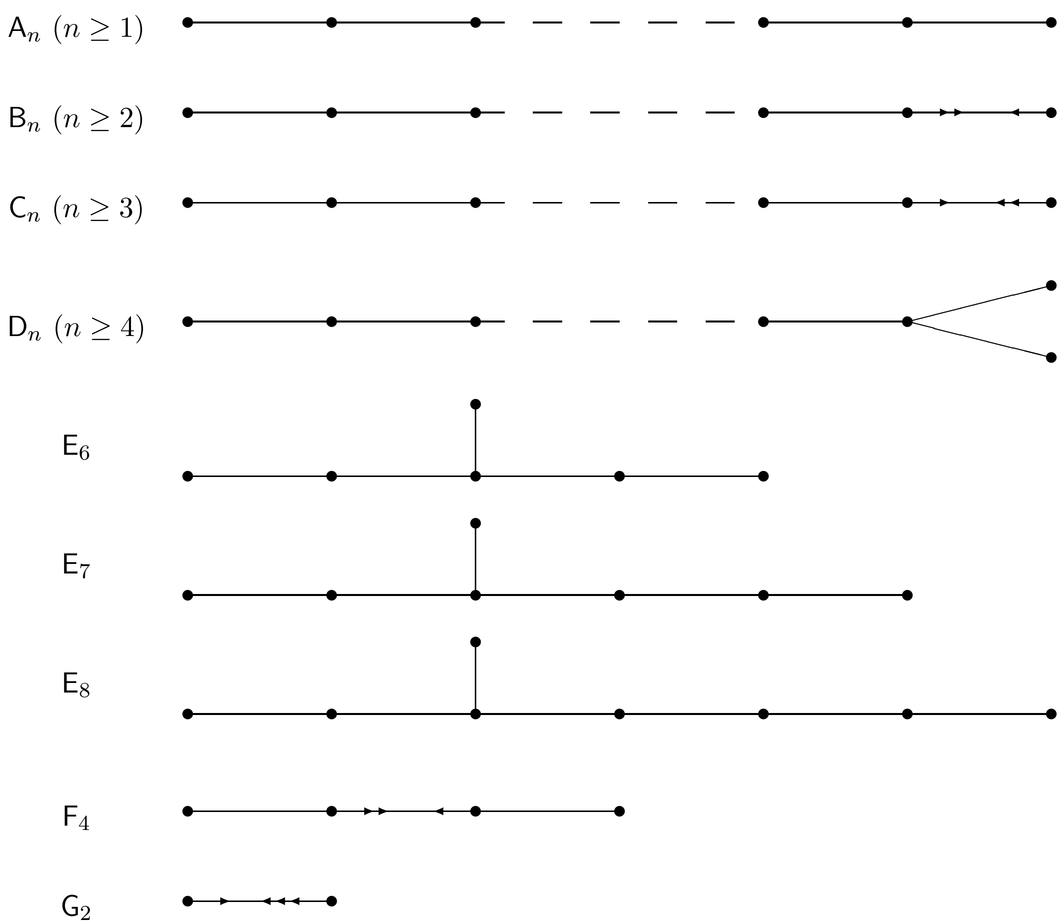
When $p = 1$ and $q = 1$ it is convenient to use the graph  to represent the GCM graph A_2. A GCM graph (Γ, M) is a *Dynkin diagram of finite type* if each connected component

Figure 1.1: Connected Dynkin diagrams of finite type.



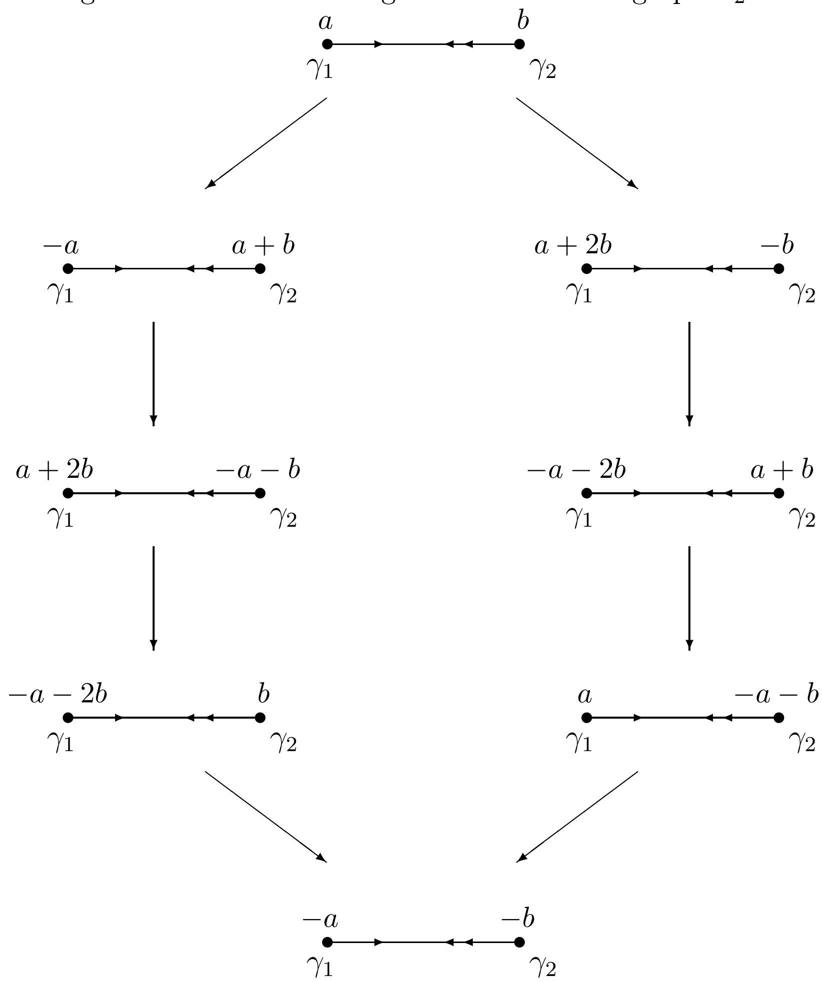
of (Γ, M) is one of the graphs of Figure 1.1. We number our nodes as in §11.4 of [Hum]. In these cases the GCMs are "Cartan" matrices.

A *position* $\lambda = (\lambda_i)_{i \in I_n}$ is an assignment of real numbers to the nodes of the GCM graph (Γ, M). The position λ is *dominant* (respectively, *strongly dominant*) if $\lambda_i \geq 0$ (resp. $\lambda_i > 0$) for all $i \in I_n$; λ is *nonzero* if at least one $\lambda_i \neq 0$. For $i \in I_n$, the *fundamental position* ω_i is the assignment of the number 1 at node γ_i and the number 0 at all other nodes. Given a position λ on a GCM graph (Γ, M), to *fire* a node γ_i is to change the number at each node γ_j of Γ by the transformation

$$\lambda_j \longmapsto \lambda_j - M_{ij}\lambda_i,$$

provided the number at node γ_i is positive; otherwise node γ_i is not allowed to be fired. Since the generalized Cartan matrix M assigns a pair of *amplitudes* (M_{ij} and M_{ji}) to each edge of the graph Γ, we sometimes refer to GCMs as *amplitude matrices.* The *numbers game* is the one-player game on a GCM graph (Γ, M) in which the player (1) Assigns an initial position to the nodes of Γ; (2) Chooses a node with a positive number and fires the node to obtain a new position; and (3) Repeats step (2) for the new position if there is at least one node with a positive number. Consider now the GCM graph B_2. As we can see in Figure 1.2, the numbers game terminates in a finite number of steps for any initial position and any legal sequence of node firings, if it is understood that the player will continue to fire as long as there is at least one node with a positive number. In general, given a position λ, a *game sequence for* λ is the (possibly empty, possibly infinite) sequence

Figure 1.2: The numbers game for the GCM graph B_2.



$(\gamma_{i_1}, \gamma_{i_2}, \ldots)$, where γ_{i_j} is the jth node that is fired in some numbers game with initial position λ. More generally, a *firing sequence* from some position λ is an initial portion of some game sequence played from λ; the phrase *legal firing sequence* is used to emphasize that all node firings in the sequence are known or assumed to be possible. Note that a game sequence $(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_l})$ is of finite length l (possibly with $l = 0$) if the number is nonpositive at each node after the lth firing; in this case we say the game sequence is *convergent* and the resulting position is the *terminal position* for the game sequence. We say a connected GCM graph (Γ, M) is *admissible* if there exists a nonzero dominant initial position with a convergent game sequence.

The following preliminary results are needed for the proofs of Propositions 2.3 and 3.1. These results also appear in [DE] and [Don2] for use in proofs of key theorems of those papers. Proofs or references for these results are also given here. Following [Erik2] and [Erik6], we say the numbers game on a GCM graph (Γ, M) is *strongly convergent* if given any initial position, any two game sequences either both diverge or both converge to the same terminal position in the same number of steps. The next result follows from Theorem 3.1 of [Erik6] (or see Theorem 3.6 of [Erik2]).

Theorem 1.1 (Eriksson's Strong Convergence Theorem) *The numbers game on a connected GCM graph is strongly convergent.*

The following weaker result also applies when the GCM graph is not connected:

Lemma 1.2 *For any GCM graph, if a game sequence for an initial position λ diverges, then all game sequences for λ diverge.*

The next result is an immediate consequence of Theorem 4.3 of [Erik2] or Theorem 4.5 of [Erik5]. Eriksson's proof of this result in [Erik2] uses only combinatorial and linear algebraic methods.

Theorem 1.3 (Eriksson's Comparison Theorem) *Given a GCM graph, suppose that a game sequence for an initial position $\lambda = (\lambda_i)_{i \in I_n}$ converges. Suppose that a position $\lambda' := (\lambda'_i)_{i \in I_n}$ has the property that $\lambda'_i \leq \lambda_i$ for all $i \in I_n$. Then some game sequence for the initial position λ' also converges.*

Let r be a positive real number. Observe that if $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ is a convergent game sequence for an initial position $\lambda = (\lambda_i)_{i \in I_n}$, then $(\gamma_{i_1}, \ldots, \gamma_{i_l})$ is a convergent game sequence for the initial position $r\lambda := (r\lambda_i)_{i \in I_n}$. This observation and Theorem 1.3 imply the following result:

Lemma 1.4 *Let $\lambda = (\lambda_i)_{i \in I_n}$ be a dominant initial position such that $\lambda_j > 0$ for some $j \in I_n$. Suppose that a game sequence for λ converges. Then some game sequence for the fundamental position ω_j also converges.*

The following is an immediate consequence of Lemmas 1.2 and 1.4:

Lemma 1.5 *A GCM graph is not admissible if for each fundamental position there is a divergent game sequence.*

2. Convergent numbers games on Dynkin diagrams of finite type

Eriksson's Strong Convergence and Comparison Theorems are key steps in our proof of Proposition 2.3. The remaining step, which accounts for most of the length of this section, is to provide convergent game sequences for numbers games played from strongly dominant positions on connected Dynkin diagrams of finite type. Finding convergent game sequences for numbers games played on Dynkin diagrams of finite type may seem like a difficult task at first, but in view of Proposition 2.3, there is no way to go wrong: any two numbers games played from the same initial position will terminate in the same finite number of steps.

A general theory connecting the numbers game and Coxeter/Weyl group actions was developed by Eriksson in [Erik2] and [Erik5]. From this theory it follows that for a numbers game played from a strongly dominant position on a Dynkin diagram of finite type, any game sequence corresponds to a reduced expression for the longest element of the corresponding Weyl group and conversely any reduced expression for the longest Weyl group element corresponds to a game sequence (see Propositions 4.1 and 4.2 of [Erik5] or Theorem 4.3.1 part (*iv*) of [BB]). Moreover, the length of the game sequence is the length of any such reduced expression and is also equal to the number of positive roots in the associated root system. For further discussion of this phenomenon, see [Don2].

For the four infinite families of connected Dynkin diagrams of finite type, the next results are proved by induction on n, the number of nodes. This is effected by observing natural "GCM subgraph" inclusions $\mathsf{A}_{n-1} \hookrightarrow \mathsf{A}_n$ $(n \geq 2)$, $\mathsf{B}_{n-1} \hookrightarrow \mathsf{B}_n$ $(n \geq 3)$, $\mathsf{C}_{n-1} \hookrightarrow \mathsf{C}_n$ $(n \geq 4)$, and $\mathsf{D}_{n-1} \hookrightarrow \mathsf{D}_n$ $(n \geq 5)$. If I'_m is a subset of the node set I_n of a GCM graph (Γ, M), then let Γ' be the subgraph of Γ with node set I'_m and the induced set of edges, and let M' be the corresponding submatrix of the amplitude matrix M; we call (Γ', M') a *GCM subgraph* of (Γ, M).

Lemma 2.1

A. *For $n \geq 2$ and for any strongly dominant position $(a_1, \ldots, a_n)$ on A_n, one can obtain the*

position $(a_1+\cdots+a_n, -a_n, \ldots, -a_3, -a_2)$ *by the sequence* $(\mathbf{s}_n,\ \mathbf{s}_{n-1},\ \ldots,\ \mathbf{s}_2)$ *of legal node firings where* $\mathbf{s}_i$ *is the subsequence* $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-1},\ \gamma_n)$ *for* $2 \leq i \leq n$.

B. *A similar statement holds for* B_n *with* $n \geq 2$*: The initial strongly dominant position is* $(a_1, \ldots, a_n)$*, and the position* $(a_1+2a_2+\cdots+2a_{n-1}+a_n, -a_2, -a_3, \ldots, -a_n)$ *is obtained by the sequence* $(\mathbf{s}_n,\ \mathbf{s}_{n-1},\ \ldots,\ \mathbf{s}_2)$ *of legal node firings where* $\mathbf{s}_i$ *is the subsequence* $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots,\ \gamma_{i+1},\ \gamma_i)$ *for* $2 \leq i \leq n-1$ *and* $\mathbf{s}_n = (\gamma_n)$.

C. *A similar statement holds for* C_n *with* $n \geq 3$*: The initial strongly dominant position is* $(a_1, \ldots, a_n)$*, and the position* $(a_1+2a_2+\cdots+2a_{n-1}+2a_n, -a_2, -a_3, \ldots, -a_n)$ *is obtained by the sequence* $(\mathbf{s}_n,\ \mathbf{s}_{n-1},\ \ldots,\ \mathbf{s}_2)$ *of legal node firings where* $\mathbf{s}_i$ *is the subsequence* $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots,\ \gamma_{i+1},\ \gamma_i)$ *for* $2 \leq i \leq n-1$ *and* $\mathbf{s}_n = (\gamma_n)$.

D. *A similar statement holds for* D_n *with* $n \geq 4$*: The initial strongly dominant position is* $(a_1, \ldots, a_n)$*. Let* $b_{n-1} := a_{n-1}$ *and* $b_n := a_n$ *when* n *is odd and where* $b_{n-1} := a_n$ *and* $b_n := a_{n-1}$ *when* n *is even. The position* $(a_1+2a_2+\cdots+2a_{n-2}+a_{n-1}+a_n, -a_2, -a_3, \ldots, -a_{n-2}, -b_{n-1}, -b_n)$ *is obtained by the sequence* $(\mathbf{s}_{n-1},\ \mathbf{s}_{n-2},\ \ldots,\ \mathbf{s}_2)$ *of legal node firings where* $\mathbf{s}_i$ *is the subsequence* $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-2},\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-2},\ \ldots,\ \gamma_{i+1},\ \gamma_i)$ *for* $2 \leq i \leq n-2$ *and* $\mathbf{s}_{n-1} = (\gamma_{n-1}, \gamma_n)$.

Proof. In case A, the result clearly holds for the two-node graph. As our induction hypothesis, assume the lemma statement holds for all type A Dynkin diagrams with fewer than n nodes. Given A_n, the GCM subgraph determined by the $n-1$ rightmost nodes is an A_{n-1} Dynkin diagram. Applying the induction hypothesis, the legal firing sequence $(\mathbf{s}_n, \ldots, \mathbf{s}_3)$ from the strongly dominant position $(a_1, a_2, \ldots, a_n)$ results in the position $(a_1, a_2+\cdots+a_n, -a_n, \ldots, -a_3)$. To this position we now apply the sequence $\mathbf{s}_2 = (\gamma_2,\ \ldots,\ \gamma_{n-1},\ \gamma_n)$. Once γ_2 is fired, then for $3 \leq i \leq n$ it is easily seen that just before γ_i is fired in the sequence $\mathbf{s}_2$ the position is $(a_1+a_2+\cdots+a_n,\ -a_n,\ -a_{n-1},\ \ldots,\ -a_{n+5-i},\ -a_{n+4-i},\ -a_2-a_3-\cdots-a_{n+3-i},\ a_2+\cdots+a_{n+2-i},\ -a_{n+2-i},\ -a_{n+1-i},\ \ldots,\ -a_4,\ -a_3)$. Then each firing in the sequence $\mathbf{s}_2$ is legal, and the resulting position is $(a_1+a_2+\cdots+a_n,\ -a_n,\ -a_{n-1},\ \ldots,\ -a_3,\ -a_2)$.

In case B, the result clearly holds for the two-node graph. As our induction hypothesis, assume the lemma statement holds for all type B Dynkin diagrams with fewer than n nodes. Given B_n, the GCM subgraph determined by the $n-1$ rightmost nodes is a B_{n-1} Dynkin diagram. Applying the induction hypothesis, the legal firing sequence $(\mathbf{s}_n,\ \ldots,\ \mathbf{s}_3)$ from the strongly dominant position $(a_1, a_2, \ldots, a_n)$ results in the position $(a_1, a_2+2a_3+\cdots+2a_{n-1}+a_n, -a_3, \ldots, -a_n)$. To this position we now apply the sequence $\mathbf{s}_2 = (\gamma_2,\ \ldots,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots\ \gamma_2)$. Once γ_2 is fired, then for $3 \leq i \leq n-1$ it is easily seen that just before γ_i is fired for the first time in the sequence $\mathbf{s}_2$ the position is $(a_1+a_2+2a_3+\cdots+2a_{n-1}+a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{i-1},\ -a_2-a_3-\cdots-a_{i-1}-2a_i-\cdots-2a_{n-1}-a_n,\ a_2+a_3+\cdots+a_{i-1}+a_i+2a_{i+1}+\cdots+2a_{n-1}+a_n,\ -a_{i+1},\ -a_{i+2},\ \ldots,\ -a_n)$. One now sees that just before γ_n is fired the position is $(a_1+a_2+2a_3+\cdots+2a_{n-1}+a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{n-1},\ -a_2-a_3-\cdots-a_{n-1}-a_n,\ 2a_2+2a_3+\cdots+2a_{n-1}+a_n)$. Now for $3 \leq i \leq n-1$, it is easily seen that just before γ_i is fired for the second time in the sequence $\mathbf{s}_2$ the position is $(a_1+a_2+2a_3+\cdots+2a_{n-1}+a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_i,\ a_2+a_3+\cdots+a_{i-1}+a_i,\ -a_2-a_3+\cdots-a_i-a_{i+1},\ -a_{i+2},\ -a_{i+3},\ \ldots,\ -a_n)$. Finally, fire γ_2 from the position $(a_1+a_2+2a_3+\cdots+2a_{n-1}+a_n,\ a_2,\ -a_2-a_3,\ -a_4,\ \ldots,\ -a_n)$. Then each firing in the sequence $\mathbf{s}_2$ is legal, and the resulting position is $(a_1+2a_2+\cdots+2a_{n-1}+a_n,\ -a_2,\ -a_3,\ \ldots,\ -a_n)$.

In case C, it is easy to confirm that the specified sequence of four legal node firings played from a strongly dominant position on the three-node graph yields the stated resulting position. As our induction hypothesis, assume the lemma statement holds for all type C Dynkin diagrams with fewer than n nodes. Given C_n, the GCM subgraph determined by the $n-1$ rightmost nodes is a C_{n-1} Dynkin diagram. Applying the induction hypothesis, the legal firing sequence $(\mathbf{s}_n, \ldots, \mathbf{s}_3)$ from the strongly dominant position $(a_1, a_2, \ldots, a_n)$ results in the position $(a_1, a_2 + 2a_3 + \cdots + 2a_n, -a_3, \ldots, -a_n)$. To this position we now apply the sequence $\mathbf{s}_2 = (\gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots\ \gamma_2)$. Once γ_2 is fired, then for $3 \leq i \leq n-1$ it is easily seen that just before γ_i is fired for the first time in the sequence $\mathbf{s}_2$ the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{i-1},\ -a_2 - a_3 - \cdots - a_{i-1} - 2a_i - \cdots - 2a_n,\ a_2 + a_3 + \cdots + a_{i-1} + a_i + 2a_{i+1} + \cdots + 2a_n,\ -a_{i+1},\ -a_{i+2},\ \ldots,\ -a_n)$. One now sees that just before γ_n is fired the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{n-1},\ -a_2 - a_3 - \cdots - a_{n-1} - 2a_n,\ a_2 + a_3 + \cdots + a_n)$. Now for $3 \leq i \leq n-1$, it is easily seen that just before γ_i is fired for the second time in the sequence $\mathbf{s}_2$ the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_i,\ a_2 + a_3 + \cdots + a_{i-1} + a_i,\ -a_2 - a_3 + \cdots - a_i - a_{i+1},\ -a_{i+2},\ -a_{i+3},\ \ldots,\ -a_n)$. Finally, fire γ_2 from the position $(a_1 + a_2 + 2a_3 + \cdots + 2a_n,\ a_2,\ -a_2 - a_3,\ -a_4,\ \ldots,\ -a_n)$. Then each firing in the sequence $\mathbf{s}_2$ is legal, and the resulting position is $(a_1 + 2a_2 + \cdots + 2a_n,\ -a_2,\ -a_3,\ \ldots,\ -a_n)$.

In case D, it is easy to confirm that the specified sequence of six legal node firings played from a strongly dominant position on the four-node graph yields the stated resulting position. As our induction hypothesis, assume the lemma statement holds for all type D Dynkin diagrams with fewer than n nodes. Given D_n, the GCM subgraph determined by the $n-1$ rightmost nodes is a D_{n-1} Dynkin diagram. Assume for the moment that n is even, so $n-1$ is odd. Applying the induction hypothesis, the legal firing sequence $(\mathbf{s}_{n-1}, \ldots, \mathbf{s}_3)$ from the strongly dominant position $(a_1, a_2, \ldots, a_n)$ results in the position $(a_1, a_2 + 2a_3 + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ -a_3,\ \ldots,\ -a_{n-2},\ -a_{n-1},\ -a_n)$. To this position we now apply the sequence $\mathbf{s}_2 = (\gamma_2, \ldots, \gamma_{n-2}, \gamma_{n-1}, \gamma_n, \gamma_{n-2}, \ldots\ \gamma_2)$. Once γ_2 is fired, then for $3 \leq i \leq n-3$ it is easily seen that just before γ_i is fired for the first time in the sequence $\mathbf{s}_2$ the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{i-1},\ -a_2 - a_3 - \cdots - a_{i-1} - 2a_i - \cdots - 2a_{n-2} - a_{n-1} - a_n,\ a_2 + a_3 + \cdots + a_{i-1} + a_i + 2a_{i+1} + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ -a_{i+1},\ -a_{i+2},\ \ldots,\ -a_{n-2},\ -a_{n-1},\ -a_n)$. One now sees that just before γ_{n-2} is fired for the first time in the sequence the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{n-3},\ -a_2 - a_3 - \cdots - a_{n-3} - 2a_{n-2} - a_{n-1} - a_n,\ a_2 + a_3 + \cdots + a_{n-2} + a_{n-1} + a_n,\ -a_{n-1},\ -a_n)$. Then just before γ_{n-1} is fired the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{n-3},\ -a_{n-2},\ -a_2 - a_3 - \cdots - a_{n-2} - a_{n-1} - a_n,\ a_2 + a_3 + \cdots + a_{n-2} + a_n,\ a_2 + a_3 + \cdots + a_{n-2} + a_{n-1})$, and just before γ_n is fired the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{n-3},\ -a_{n-2},\ -a_{n-1},\ -a_2 - a_3 - \cdots - a_{n-2} - a_n,\ a_2 + a_3 + \cdots + a_{n-2} + a_{n-1})$. So just before γ_{n-2} is fired for the second time in the sequence $\mathbf{s}_2$ the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_{n-3},\ -a_{n-2},\ a_2 + a_3 + \cdots + a_{n-2},\ -a_2 - a_3 - \cdots - a_{n-2} - a_n,\ -a_2 - a_3 - \cdots - a_{n-2} - a_{n-1})$. Now for $3 \leq i \leq n$, it is easily seen that just before γ_i is fired for the second time in the sequence $\mathbf{s}_2$ the position is $(a_1 + a_2 + 2a_3 + \cdots + 2a_n,\ -a_3,\ -a_4,\ \ldots,\ -a_i,\ a_2 + a_3 + \cdots + a_{i-1} + a_i,\ -a_2 - a_3 + \cdots - a_i - a_{i+1},\ -a_{i+2},\ -a_{i+3},\ \ldots,\ -a_{n-2},\ -a_n,\ -a_{n-1})$. Finally, fire γ_2 from the position $(a_1 + a_2 + 2a_3 + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ a_2,\ -a_2 - a_3,\ -a_4,\ \ldots,\ -a_{n-2},\ -a_n,\ -a_{n-1})$. Then each firing in the sequence $\mathbf{s}_2$ is legal, and the resulting position is $(a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1} + a_n,\ -a_2,\ -a_3,\ \ldots,\ a_{n-2},\ -a_n,\ -a_{n-1})$. When n is odd, the argument is entirely similar. $\square$

From this we immediately obtain the following:

Lemma 2.2

A. *For any positive integer n and for any strongly dominant position $(a_1, \ldots, a_n)$ on A_n, one can obtain the position $(-a_n, \ldots, -a_2, -a_1)$ by the sequence $(\mathbf{s}_n,\ \mathbf{s}_{n-1},\ \ldots,\ \mathbf{s}_1)$ of legal node firings where $\mathbf{s}_i$ is the subsequence $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-1},\ \gamma_n)$ for $1 \leq i \leq n$.*

B. *A similar statement holds for B_n with $n \geq 2$: The initial strongly dominant position is $(a_1, \ldots, a_n)$, and the position $(-a_1, -a_2, \ldots, -a_n)$ is obtained by the sequence $(\mathbf{s}_n,\ \mathbf{s}_{n-1},\ \ldots,\ \mathbf{s}_1)$ of legal node firings where $\mathbf{s}_i$ is the subsequence $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots,\ \gamma_{i+1},\ \gamma_i)$ for $1 \leq i \leq n-1$ and $\mathbf{s}_n = (\gamma_n)$.*

C. *A similar statement holds for C_n with $n \geq 3$: The initial strongly dominant position is $(a_1, \ldots, a_n)$, and the position $(-a_1, -a_2, \ldots, -a_n)$ is obtained by the sequence $(\mathbf{s}_n,\ \mathbf{s}_{n-1},\ \ldots,\ \mathbf{s}_1)$ of legal node firings where $\mathbf{s}_i$ is the subsequence $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots,\ \gamma_{i+1},\ \gamma_i)$ for $1 \leq i \leq n-1$ and $\mathbf{s}_n = (\gamma_n)$.*

D. *A similar statement holds for D_n with $n \geq 4$: The initial strongly dominant position is $(a_1, \ldots, a_n)$. Let $b_{n-1} := a_{n-1}$ and $b_n := a_n$ when n is even and where $b_{n-1} := a_n$ and $b_n := a_{n-1}$ when n is odd. The position $(a_1 + 2a_2 + \cdots + 2a_{n-2} + a_{n-1} + a_n, -a_2, -a_3, \ldots, -a_{n-2}, -b_{n-1}, -b_n)$ is obtained by the sequence $(\mathbf{s}_{n-1},\ \mathbf{s}_{n-2},\ \ldots,\ \mathbf{s}_1)$ of legal node firings where $\mathbf{s}_i$ is the subsequence $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-2},\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-2},\ \ldots,\ \gamma_{i+1},\ \gamma_i)$ for $1 \leq i \leq n-2$ and $\mathbf{s}_{n-1} = (\gamma_{n-1}, \gamma_n)$.*

Proof. View A_n (respectively B_n, C_n, D_n) as a GCM subgraph of A_{n+1} (respectively B_{n+1}, C_{n+1}, D_{n+1}) by adding a node "to the left" of γ_1. Conclude by applying Lemma 2.1. □

Proposition 2.3 *A connected Dynkin diagram (Γ, M) of finite type is admissible. Moreover, for any initial position on (Γ, M), all game sequences converge to the same terminal position in the same finite number of steps.*

Proof. The Strong Convergence Theorem shows that if a game sequence for some initial position λ on (Γ, M) converges, then all game sequences from λ converge to the same terminal position in the same finite number of steps. Then in light of The Comparison Theorem, it suffices to show that for any strongly dominant initial position on (Γ, M), there is a convergent game sequence.

For the four infinite families (A_n, B_n, C_n, and D_n), this is handled by Lemma 2.2. For the exceptional graphs (E_6, E_7, E_8, F_4, and G_2) this can be checked by hand (requiring 36, 63, 120, 24, and 6 firings respectively). For G_2, begin with strongly dominant position $\lambda = (a, b)$ with positive number a on node γ_1 and positive number b on γ_2. The following sequence of six node firings is easily seen to be legal: $(\gamma_1,\ \gamma_2,\ \gamma_1,\ \gamma_2,\ \gamma_1,\ \gamma_2)$. The resulting position is $(-a, -b)$. For F_4, begin with strongly dominant position (a, b, c, d). Play the numbers game from this initial position to see that the following sequence of 24 node firings is legal: $(\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_3,\ \gamma_2,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_2,\ \gamma_3,\ \gamma_2,\ \gamma_1,\ \gamma_4,\ \gamma_3,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_2,\ \gamma_1,\ \gamma_3,\ \gamma_2,\ \gamma_3)$. The resulting position is $(-a, -b, -c, -d)$. For E_6, begin with strongly dominant position (a, b, c, d, e, f). Play the numbers game from this initial position to see that the following sequence of 36 node firings is legal: $(\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_3,\ \gamma_2,\ \gamma_1,\ \gamma_4,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_4,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_1,\ \gamma_3,\ \gamma_5,\ \gamma_6,\ \gamma_4,\ \gamma_5,\ \gamma_4,\ \gamma_2,\ \gamma_3,\ \gamma_1,\ \gamma_4,\ \gamma_3,\ \gamma_1,\ \gamma_5,\ \gamma_4,\ \gamma_2,\ \gamma_6,\ \gamma_5,\ \gamma_4,\ \gamma_3,\ \gamma_1)$. The resulting position is $(-f, -b, -e, -d, -c, -a)$. For E_7, begin with strongly dominant position (a, b, c, d, e, f, g). Play the numbers game from this initial position to see that the following sequence of 63 node firings is legal (the first 36 of these node firings are exactly the previous game sequence played on the E_6 subgraph): $(\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_3,\ \gamma_2,\ \gamma_1,\ \gamma_4,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_4,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_1,$

γ_3, γ_5, γ_6, γ_4, γ_5, γ_4, γ_2, γ_3, γ_1, γ_4, γ_3, γ_1, γ_5, γ_4, γ_2, γ_6, γ_5, γ_4, γ_3, γ_1, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_1, γ_4, γ_3, γ_5, γ_4, γ_2, γ_6, γ_5, γ_4, γ_3, γ_1, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7). The resulting position is $(-a,-b,-c,-d,-e,-f,-g)$. For E_8, begin with strongly dominant position (a,b,c,d,e,f,g,h). Play the numbers game from this initial position to see that the following sequence of 120 node firings is legal (the first 63 of these node firings are exactly the previous game sequence played on the E_7 subgraph): (γ_1, γ_2, γ_3, γ_4, γ_3, γ_2, γ_1, γ_4, γ_3, γ_4, γ_5, γ_4, γ_2, γ_3, γ_4, γ_1, γ_3, γ_5, γ_6, γ_4, γ_5, γ_4, γ_2, γ_3, γ_1, γ_4, γ_3, γ_1, γ_5, γ_4, γ_2, γ_6, γ_5, γ_4, γ_3, γ_1, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_1, γ_4, γ_3, γ_5, γ_4, γ_2, γ_6, γ_5, γ_4, γ_3, γ_1, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7, γ_8, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_1, γ_4, γ_3, γ_5, γ_4, γ_2, γ_6, γ_5, γ_4, γ_3, γ_1, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7, γ_8, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_1, γ_4, γ_3, γ_5, γ_4, γ_2, γ_6, γ_7, γ_5, γ_6, γ_4, γ_3, γ_1, γ_5, γ_4, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7, γ_8). The resulting position is $(-a,-b,-c,-d,-e,-f,-g,-h)$. $\square$

Remark 2.4 As can be seen from Lemma 2.2 and Proposition 2.3, the length of any game sequence played from any strongly dominant initial position on A_n (respectively B_n, C_n, D_n) is $\frac{n(n+1)}{2}$ (respectively n^2, n^2, $n(n-1)$). Similarly, from the statement and proof of Proposition 2.3 it follows that the length of any game sequence played from any strongly dominant initial position on E_6 (respectively E_7, E_8, F_4, G_2) is 36 (respectively 63, 120, 24, 6). $\square$

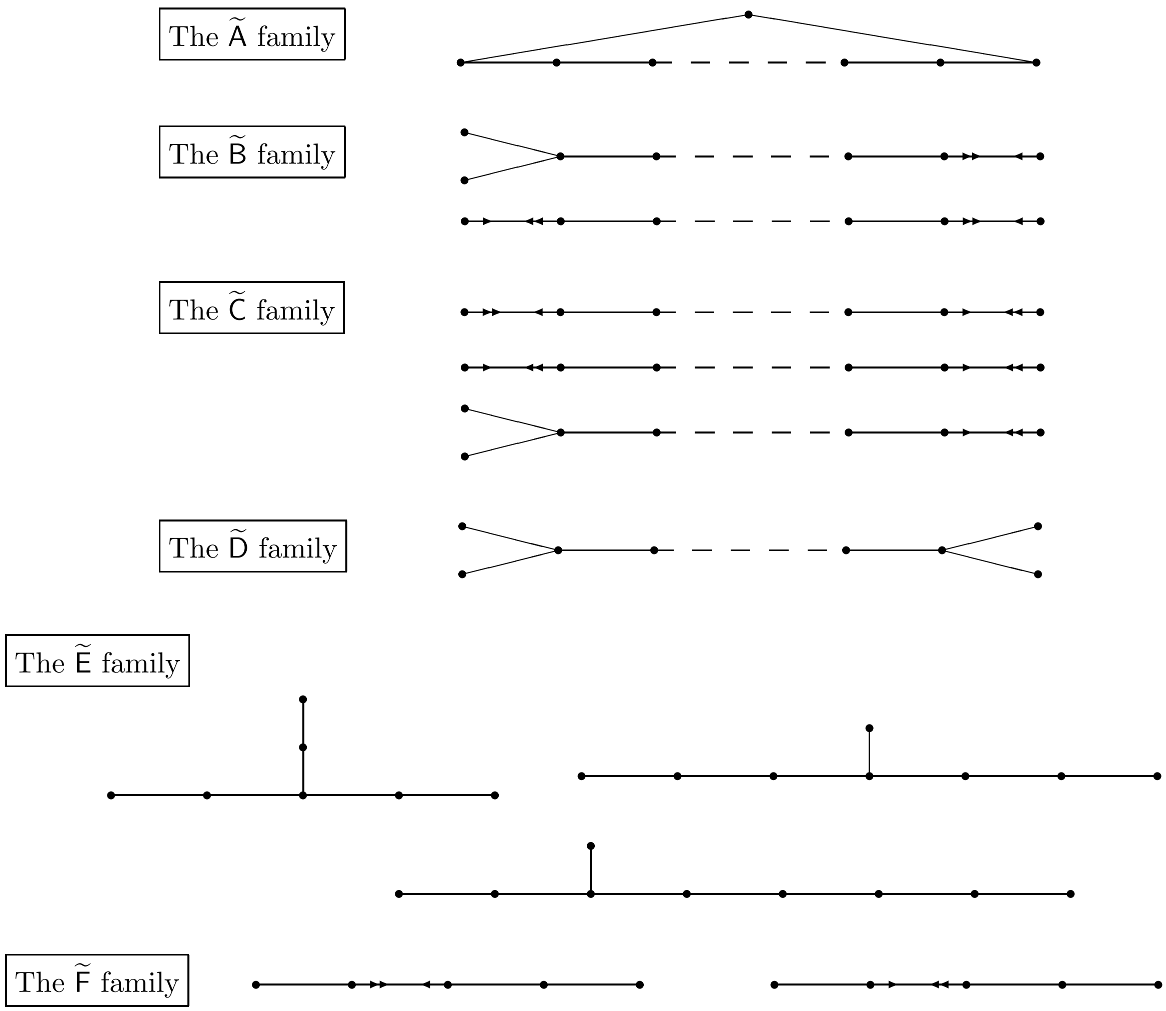
3. Divergent games for some families of graphs

Proposition 3.1 *The connected GCM graphs of Figure 3.1 are not admissible.*

Proof. By Lemma 1.5 it suffices to show that for each graph in Figure 3.1 and for each fundamental position, there is a divergent game sequence. In each case we exhibit a divergent game sequence which is a simple pattern of node firings. Remarkably, in all cases trial and error quickly lead us to these patterns. Our goal in this proof is not to develop any general theory for finding divergent game sequences for these cases, but rather to show that such game sequences can be found and presented in an elementary (though sometimes tedious) manner. The $\widetilde{\mathsf{A}}$, $\widetilde{\mathsf{B}}$, $\widetilde{\mathsf{C}}$, $\widetilde{\mathsf{D}}$, $\widetilde{\mathsf{E}}$, and $\widetilde{\mathsf{F}}$ cases are handled using a common line of reasoning: A sequence of legal node firings is applied to a position whose numbers are linear expressions in an index variable k. It is then observed that the numbers for the resulting position are linear expressions of the same form with respect to the variable $k+1$ and that the firing sequence can be repeated. The $\widetilde{\mathsf{G}}$ cases and the families of small cycles are handled using a variation of this kind of argument: A sequence of legal node firings is applied to a generic position satisfying certain inequalities, and it is shown that the resulting position also satisfies these inequalities so that the firing sequence can be repeated. Each paragraph in what follows demonstrates inadmissibility for some graph in our list. Our case-analysis argument is lengthy in part because we have tried to make each paragraph reasonably self-contained.

$\boxed{\text{The } \widetilde{\mathsf{A}} \text{ family}}$ The infinite $\widetilde{\mathsf{A}}$ family of GCM graphs of Figure 3.1 is the family of cycles with amplitude products of unity on all edges. Such cycles were in fact the graphs that motivated Mozes' study of the numbers game in [Moz]. The argument we give here demonstrating inadmissibility for each graph in this family is a special case of the proof of Lemma 3.1 of [Don2]. For an n-node graph in the $\widetilde{\mathsf{A}}$ family (we take $n \geq 3$), number the top node γ_1 and the remaining nodes $\gamma_2,\ldots,\gamma_n$ in succession in the clockwise order around the cycle. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. By symmetry, it suffices to do so for the fundamental position $\omega_1 = (1,0,\ldots,0)$. This is the $k=0$

Figure 3.1: Some connected GCM graphs that are not admissible.
(Figure continues on the next page.)



version of the position $(2k+1, -k, 0, \ldots, 0, -k)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1,\ \gamma_2,\ \ldots\ ,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots\ ,\ \gamma_3,\ \gamma_2)$. This sequence results in the position $(2(k+1)+1, -(k+1), 0, \ldots, 0, -(k+1))$. This gives the desired divergent game sequence. We conclude that any such GCM graph is inadmissible.

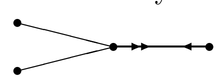
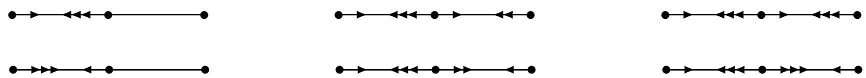
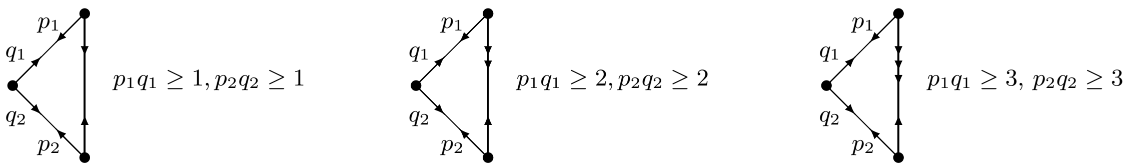
The $\widetilde{\mathsf{B}}$ family First, we show why  is not admissible. Label the leftmost nodes as γ_1 and γ_2, the middle node as γ_3, and the rightmost node as γ_4. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. From the fundamental position $\omega_1 = (1, 0, 0, 0)$, play the (legal) sequence $(\gamma_1,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_3)$ to obtain the position $(2, 1, -2, 2)$. This is the $k = 0$ version of the position $(k+2, k+1, -2(k+1), 2(k+1))$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1,\ \gamma_2,\ \gamma_4,\ \gamma_3,\ \gamma_1,\ \gamma_2,\ \gamma_4,\ \gamma_3)$. This sequence results in the position $((k+1)+2, (k+1)+1, -2[(k+1)+1], 2[(k+1)+1])$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_2. The fundamental position $\omega_3 = (0, 0, 1, 0)$ is the $k = 0$ version of the position $(-2k, -2k, 2k+1, 0)$. From any such position with $k \geq 0$, the following

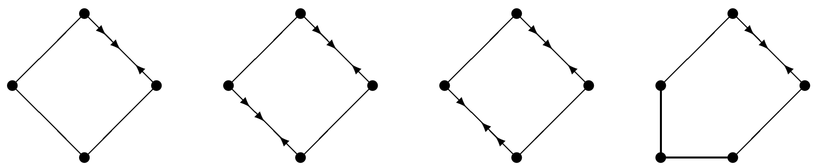
Figure 3.1 (continued): Some connected GCM graphs that are not admissible.

The $\widetilde{\mathsf{G}}$ family

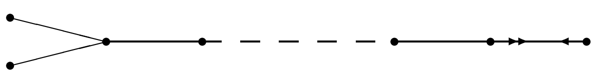


Families of small cycles





sequence of node firings is easily seen to be legal: (γ_3, γ_4, γ_3, γ_2, γ_1). This sequence results in the position $(-2(k+1), -2(k+1), 2(k+1)+1, 0)$. The fundamental position $\omega_4 = (0, 0, 0, 1)$ is the $k = 0$ version of the position $(0, 0, -k, 2k+1)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_4, γ_3, γ_2, γ_1, γ_3). This sequence results in the position $(0, 0, -(k+1), 2(k+1)+1)$.

Next, we show why  is not admissible when the graph has $n \geq 5$ nodes. Label the leftmost nodes as γ_1 and γ_2, and label the remaining nodes in succession from left to right as γ_3, ... , γ_{n-1}, γ_n. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(2k+1, -2k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_1, γ_3, γ_4, ... , γ_{n-1}, γ_n, γ_{n-1}, ... , γ_3, γ_2, γ_1, γ_3, γ_4, ... , γ_{n-1}, γ_n, γ_{n-1}, ... , γ_3, γ_2). This sequence results in the position $(2(k+1)+1, -2(k+1), 0, \ldots, 0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_2. The fundamental position $\omega_3 = (0, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(-2k, -2k, 2k+1, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_3, γ_4, ... , γ_{n-1}, γ_n, γ_{n-1}, ... , γ_3, γ_2, γ_1). This sequence results in the position $(-2(k+1), -2(k+1), 2(k+1)+1, 0, \ldots, 0)$. For $4 \leq i \leq n-1$, any fundamental position $\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(0, \ldots, 0, -2k, 2k+1, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_i, γ_{i+1}, ... , γ_{n-1}, γ_n, γ_{n-1}, ... , γ_{i+1}, γ_i, γ_{i-1}, ... , γ_3, γ_2, γ_1, γ_3, ... , γ_{i-1}). This sequence results in the position $(0, \ldots, 0, -2(k+1), 2(k+1)+1, 0, \ldots, 0)$. The fundamental position $\omega_n = (0, \ldots, 0, 1)$ is the $k = 0$ version of the position $(0, \ldots, 0, -k, 2k+1)$. From any

such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \ldots, \gamma_{n-1})$. This sequence results in the position $(0, \ldots, 0, -(k+1), 2(k+1)+1)$.

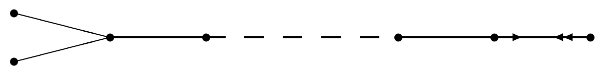
Next, we show why is not admissible. Label the nodes as γ_1, γ_2, and γ_3 from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, 0)$ is the $k = 0$ version of the position $(2k+1, -k, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1, \gamma_2, \gamma_3, \gamma_2)$. This sequence results in the position $(2(k+1)+1, -(k+1), 0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_3. The fundamental position $\omega_2 = (0, 1, 0)$ is the $k = 0$ version of the position $(-4k, 2k+1, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_2, \gamma_3, \gamma_2, \gamma_1)$. This sequence results in the position $(-4(k+1), 2(k+1)+1, 0)$.

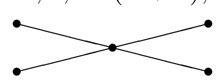
We finish the $\widetilde{\mathsf{B}}$ family by showing why is not admissible when the graph has $n \geq 4$ nodes. Label the nodes as $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$, and γ_n from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(2k+1, -k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)$. This sequence results in the position $(2(k+1)+1, -2(k+1), 0, \ldots, 0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_n. For $2 \leq i \leq n-1$, any fundamental position $\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(0, \ldots, 0, 2k+1, -2k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_i, \gamma_{i-1}, \ldots, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+2}, \gamma_{i+1})$. This sequence results in the position $(0, \ldots, 0, 2(k+1)+1, -2(k+1), 0, \ldots, 0)$.

$\boxed{\text{The } \widetilde{\mathsf{C}} \text{ family}}$ First, we show why is not admissible when the graph has $n \geq 3$ nodes. (Since firing the middle node in the $n = 3$ case is comparable to firing either γ_2 or γ_{n-1} in the $n \geq 4$ cases, then the $n = 3$ case does not need to be considered separately here.) Label the nodes as $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$, and γ_n from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(2k+1, -2k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)$. This sequence results in the position $(2(k+1)+1, -2(k+1), 0, \ldots, 0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_n. For $2 \leq i \leq n-1$, any fundamental position $\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(0, \ldots, 0, 2k+1, -2k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_i, \gamma_{i-1}, \ldots, \gamma_2, \gamma_1, \gamma_2, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_{i+2}, \gamma_{i+1})$. This sequence results in the position $(0, \ldots, 0, 2(k+1)+1, -2(k+1), 0, \ldots, 0)$.

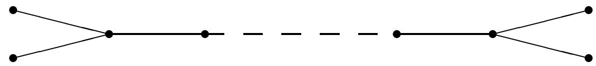
Next, we show why is not admissible when the graph has $n \geq 3$ nodes. (Since firing the middle node in the $n = 3$ case is comparable to firing γ_2 in the $n \geq 4$ cases, then the $n = 3$ case does not need to be considered separately here.) Label the nodes as $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$, and γ_n from left to right. For each fundamental position, we

exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(2k+1, -k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1,\ \gamma_2,\ \ldots\ ,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots\ ,\ \gamma_3,\ \gamma_2)$. This sequence results in the position $(2(k+1)+1, -(k+1), 0, \ldots, 0)$. For $2 \leq i \leq n-1$, any fundamental position $\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(0, \ldots, 0, 2k+1, -2k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_i,\ \gamma_{i-1},\ \ldots\ ,\ \gamma_2,\ \gamma_1,\ \gamma_2,\ \ldots\ ,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots\ ,\ \gamma_{i+2},\ \gamma_{i+1})$. This sequence results in the position $(0, \ldots, 0, 2(k+1)+1, -2(k+1), 0, \ldots, 0)$. The fundamental position $\omega_n = (0, \ldots, 0, 1)$ is the $k = 0$ version of the position $(0, \ldots, 0, -2k, 2k+1)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_n,\ \gamma_{n-1},\ \ldots\ ,\ \gamma_2,\ \gamma_1,\ \gamma_2,\ \ldots\ ,\ \gamma_{n-2},\ \gamma_{n-1})$. This sequence results in the position $(0, \ldots, 0, -2(k+1), 2(k+1)+1)$.

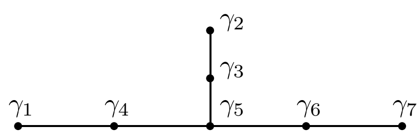
We finish the $\widetilde{\mathsf{C}}$ family by showing why  is not admissible when the graph has $n \geq 4$ nodes. (Since firing the middle node in the $n = 4$ case is comparable to firing γ_3 in the $n \geq 5$ cases, then the $n = 4$ case does not need to be considered separately here.) Label the leftmost nodes as γ_1 and γ_2, and label the remaining nodes in succession from left to right as $\gamma_3,\ \ldots\ ,\ \gamma_{n-1},\ \gamma_n$. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(2k+1, -2k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1, \gamma_3, \gamma_4, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2, \gamma_1, \gamma_3, \gamma_4, \ldots, \gamma_{n-1}, \gamma_n, \gamma_{n-1}, \ldots, \gamma_3, \gamma_2)$. This sequence results in the position $(2(k+1)+1, -2(k+1), 0, \ldots, 0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_2. The fundamental position $\omega_3 = (0, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(-2k, -2k, 2k+1, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_3,\ \gamma_4,\ \ldots\ ,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots\ ,\ \gamma_3,\ \gamma_2,\ \gamma_1)$. This sequence results in the position $(-2(k+1), -2(k+1), 2(k+1)+1, 0, \ldots, 0)$. For $4 \leq i \leq n$, any fundamental position $\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(0, \ldots, 0, -2k, 2k+1, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_i,\ \gamma_{i+1},\ \ldots\ ,\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-1},\ \ldots\ ,\ \gamma_{i+1},\ \gamma_i,\ \gamma_{i-1},\ \ldots\ ,\ \gamma_3,\ \gamma_2,\ \gamma_1,\ \gamma_3,\ \ldots\ ,\ \gamma_{i-1})$. This sequence results in the position $(0, \ldots, 0, -2(k+1), 2(k+1)+1, 0, \ldots, 0)$.

$\boxed{\text{The } \widetilde{\mathsf{D}} \text{ family}}$ First, we show why the five-node graph  is not admissible. Label the leftmost nodes as γ_1 and γ_2, label the middle node as γ_3, and label the rightmost nodes as γ_4 and γ_5. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. From the fundamental position $\omega_1 = (1, 0, 0, 0, 0)$, play the (legal) sequence $(\gamma_1,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3)$ to obtain the position $(2, 1, -2, 1, 1)$. This is the $k = 0$ version of the position $(k+2, k+1, -2(k+1), k+1, k+1)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_1,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3)$. This sequence results in the position $((k+1)+2, (k+1)+1, -2[(k+1)+1], (k+1)+1, (k+1)+1)$. By symmetry, we also obtain divergent game sequences from the fundamental positions ω_2, ω_4, and ω_5. The fundamental position $\omega_3 = (0, 0, 1, 0, 0)$ is the $k = 0$ version of the position $(-k, -k, 2k+1, -k, -k)$. From any such position with $k \geq 0$, the following sequence of

node firings is easily seen to be legal: $(\gamma_3,\ \gamma_1,\ \gamma_2,\ \gamma_4,\ \gamma_5)$. This sequence results in the position $(-(k+1), -(k+1), 2(k+1)+1, -(k+1), -(k+1))$.

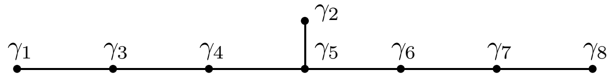
We finish the $\widetilde{\mathsf{D}}$ family by showing why  is not admissible when the graph has $n \geq 6$ nodes. Label the leftmost nodes as γ_1 and γ_2, label the "isthmus" nodes in succession from left to right as $\gamma_3, \ldots, \gamma_{n-2}$, and label the rightmost nodes as γ_{n-1} and γ_n. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(2k+1, -2k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1,\ \gamma_3,\ \gamma_4,\ \ldots,\ \gamma_{n-2},\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-2},\ \ldots,\ \gamma_4,\ \gamma_3,\ \gamma_2,\ \gamma_1,\ \gamma_3,\ \gamma_4,\ \ldots,\ \gamma_{n-2},\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-2},\ \ldots,\ \gamma_4,\ \gamma_3,\ \gamma_2)$. This sequence results in the position $(2(k+1)+1, -2(k+1), 0, \ldots, 0)$. By symmetry, we also obtain divergent game sequences from the fundamental positions ω_2, ω_{n-1}, and ω_n. The fundamental position $\omega_3 = (0, 0, 1, 0 \ldots, 0)$ is the $k = 0$ version of the position $(-2k, -2k, 2k+1, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_3,\ \gamma_4,\ \ldots,\ \gamma_{n-2},\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-2},\ \ldots,\ \gamma_4,\ \gamma_3,\ \gamma_2,\ \gamma_1)$. This sequence results in the position $(-2(k+1), -2(k+1), 2(k+1)+1, 0, \ldots, 0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_{n-2}. For $4 \leq i \leq n-2$, any fundamental position $\omega_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the $k = 0$ version of the position $(0, \ldots, 0, -k, 2k+1, -k, 0, \ldots, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_i,\ \gamma_{i+1},\ \ldots,\ \gamma_{n-2},\ \gamma_{n-1},\ \gamma_n,\ \gamma_{n-2},\ \ldots,\ \gamma_{i+1},\ \gamma_{i-1},\ \ldots,\ \gamma_3,\ \gamma_2,\ \gamma_1,\ \gamma_3,\ \ldots,\ \gamma_{i-1})$. This sequence results in the position $(0, \ldots, 0, -(k+1), 2(k+1)+1, -(k+1), 0, \ldots, 0)$.

$\boxed{\text{The } \widetilde{\mathsf{E}} \text{ family}}$ First, we show why



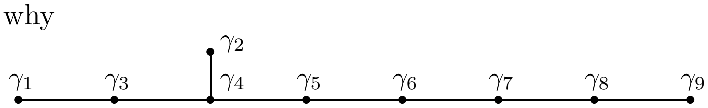
is not admissible. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1, 0, 0, 0, 0, 0, 0)$ is the $k = 0$ version of the position $(2k+1, 0, 0, -k, 0, 0, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_1,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_6,\ \gamma_5,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_6,\ \gamma_7,\ \gamma_6,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_6,\ \gamma_5,\ \gamma_4)$. This results in the position $(2(k+1)+1, 0, 0, -(k+1), 0, 0, 0)$. By symmetry, we also obtain divergent game sequences from the fundamental positions ω_2 and ω_7. The fundamental position $\omega_4 = (0, 0, 0, 1, 0, 0, 0)$ is the $k = 0$ version of the position $(-4k, 0, 0, 2k+1, 0, 0, 0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: $(\gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_6,\ \gamma_5,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_6,\ \gamma_7,\ \gamma_6,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_6,\ \gamma_5,\ \gamma_4,\ \gamma_1)$. This results in the position $(-4(k+1), 0, 0, 2(k+1)+1, 0, 0, 0)$. By symmetry, we also obtain divergent game sequences from the fundamental positions ω_3 and ω_6. The fundamental position $\omega_5 = (0, 0, 0, 0, 1, 0, 0)$ is the $k = 0$ version of the position $(-6k, 0, 0, -6k, 6k+1, 0, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: $(\gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_6,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_6,\ \gamma_5,\ \gamma_7,\ \gamma_6,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_6,\ \gamma_5,\ \gamma_4,\ \gamma_7,\ \gamma_6,\ \gamma_5,\ \gamma_1,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_6,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_6,\ \gamma_5,\ \gamma_7,\ \gamma_6,\ \gamma_5,\ \gamma_3,\ \gamma_2,\ \gamma_4,\ \gamma_5,\ \gamma_3,\ \gamma_6,\ \gamma_5,\ \gamma_4,\ \gamma_7,\ \gamma_6,\ \gamma_5,\ \gamma_1,\ \gamma_4)$. This results in the position $(-6(k+1), 0, 0, -6(k+1), 6(k+1)+1, 0, 0)$.

Next, we show why



is not admissible. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1,0,0,0,0,0,0,0)$ is the $k=0$ version of the position $(2k+1, 0, -k, 0,0,0,0,0)$. From any such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: $(\gamma_1, \gamma_3, \gamma_4, \gamma_5, \gamma_2, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3)$. This results in the position $(2(k+1)+1, 0, -(k+1), 0,0,0,0,0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_8. From the fundamental position $\omega_2 = (0,1,0,0,0,0,0,0)$, play the (legal) sequence $(\gamma_2, \gamma_5, \gamma_4, \gamma_3, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_1, \gamma_3, \gamma_4, \gamma_5)$ to obtain the position $(0,3,0,0,-4,4,0,0)$. This is the $k=0$ version of the position $(0, 2k+3, 0, 0, -2k-4, 2k+4, 0, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: $(\gamma_2, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1, \gamma_3, \gamma_4, \gamma_5)$. This results in the position $(0, 2(k+1)+3, 0, 0, -2(k+1)-4, 2(k+1)+4, 0, 0)$. The fundamental position $\omega_3 = (0,0,1,0,0,0,0,0)$ is the $k=0$ version of the position $(-4k, 0, 2k+1, 0,0,0,0,0)$. From any such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: $(\gamma_3, \gamma_4, \gamma_5, \gamma_2, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1)$. This results in the position $(-4(k+1), 0, 2(k+1)+1, 0,0,0,0,0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_7. From the fundamental position $\omega_4 = (0,0,0,1,0,0,0,0)$, play the (legal) sequence $(\gamma_4, \gamma_3, \gamma_5, \gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_7, \gamma_6, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_1, \gamma_3)$ to obtain the position $(0,0,-6,5,0,0,0,0)$. This is the $k=0$ version of the position $(-3k, 0, -3k-6, 3k+5, 0,0,0,0)$. From any such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: $(\gamma_4, \gamma_5, \gamma_2, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1, \gamma_3)$. This results in the position $(-3(k+1), 0, -3(k+1)-6, 3(k+1)+5, 0,0,0,0)$. By symmetry, we also obtain a divergent game sequence from the fundamental position ω_6. From the fundamental position $\omega_5 = (0,0,0,0,1,0,0,0)$, play the (legal) sequence $(\gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_5, \gamma_1, \gamma_3, \gamma_4)$ to obtain the position $(0,0,0,-8,7,0,0,0)$. This is the $k=0$ version of the position $(-4k, 0, 0, -4k-8, 4k+7, 0,0,0)$. From any such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: $(\gamma_5, \gamma_2, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_7, \gamma_6, \gamma_5, \gamma_2, \gamma_4, \gamma_3, \gamma_5, \gamma_4, \gamma_6, \gamma_5, \gamma_2, \gamma_7, \gamma_6, \gamma_5, \gamma_4, \gamma_3, \gamma_1, \gamma_3, \gamma_4)$. This results in the position $(-4(k+1), 0, 0, -4(k+1)-8, 4(k+1)+7, 0,0,0)$.

We finish the $\widetilde{\mathsf{E}}$ family by showing why



is not admissible. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1,0,0,0,0,0,0,0,0)$ is the $k=0$ version of the position $(2k+1, -k, 0,0,0,0,0,0,0)$. From any

such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: (γ_1, γ_3, γ_4, γ_2, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_2, γ_7, γ_6, γ_5, γ_4, γ_3, γ_8, γ_7, γ_6, γ_5, γ_4, γ_2, γ_9, γ_8, γ_7, γ_6, γ_5, γ_4, γ_3). This results in the position $(2(k+1)+1, -(k+1), 0, 0, 0, 0, 0, 0, 0)$. The fundamental position $\omega_2 = (0, 1, 0, 0, 0, 0, 0, 0, 0)$ is the $k = 0$ version of the position $(0, 3k+1, 0, -k, 0, 0, -k, 0, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence $\mathbf{s}$ of node firings is legal when played twice in a row: $\mathbf{s} :=$ (γ_2, γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6, γ_9, γ_8, γ_7). Playing $\mathbf{s}$ twice in a row results in the position $(0, 3(k+1)+1, 0, -(k+1), 0, 0, -(k+1), 0, 0)$. From the fundamental position $\omega_3 = (0, 0, 1, 0, 0, 0, 0, 0, 0)$, play the (legal) sequence (γ_3, γ_1, γ_4, γ_3, γ_5, γ_4, γ_6, γ_5, γ_7, γ_6, γ_8, γ_7, γ_9, γ_8) to obtain the position $(0, 2, 0, 0, 0, 0, 0, -1, 0)$. This is the $k = 0$ version of the position $(0, 6k+2, 0, -3k, 0, 0, -3k, -1, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence $\mathbf{s}$ of node firings is legal when played three times in a row: $\mathbf{s} :=$ (γ_2, γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6, γ_9, γ_8, γ_7). Playing $\mathbf{s}$ three times in a row results in the position $(0, 6(k+1)+2, 0, -3(k+1), 0, 0, -3(k+1), -1, 0)$. The fundamental position ω_4 is the $k = 0$ version of the position $(0, -3k, 0, 2k+1, 0, 0, -k, 0, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: (γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6, γ_9, γ_8, γ_7, γ_2). This results in the position $(0, -3(k+1), 0, 2(k+1)+1, 0, 0, -(k+1), 0, 0)$. From the fundamental position $\omega_5 = (0, 0, 0, 0, 1, 0, 0, 0, 0)$, play the (legal) sequence (γ_5, γ_4, γ_3, γ_1, γ_6, γ_5, γ_4, γ_3, γ_7, γ_6, γ_5, γ_4, γ_8, γ_7, γ_6, γ_5, γ_9, γ_8, γ_7, γ_6) to obtain the position $(0, 3, 0, 0, 0, -1, 0, 0, 0)$. This is the $k = 0$ version of the position $(0, 15k+3, 0, -5k, 0, -1, -5k, 0, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence $\mathbf{s}$ of node firings is legal when played six times in a row: $\mathbf{s} :=$ (γ_2, γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6, γ_9, γ_8, γ_7). Playing $\mathbf{s}$ six times in a row results in the position $(0, 15(k+1)+3, 0, -5(k+1), 0, -1, -5(k+1), 0, 0)$. From the fundamental position $\omega_6 = (0, 0, 0, 0, 0, 1, 0, 0, 0)$, play the (legal) sequence (γ_6, γ_5, γ_4, γ_3, γ_1, γ_7, γ_6, γ_5, γ_4, γ_3, γ_8, γ_7, γ_6, γ_5, γ_4, γ_9, γ_8, γ_7, γ_6, γ_5) to obtain the position $(0, 3, 0, 0, -1, 0, 0, 0, 0)$. This is the $k = 0$ version of the position $(0, 12k+3, 0, -4k, -1, 0, -4k, 0, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence $\mathbf{s}$ of node firings is legal when played six times in a row: $\mathbf{s} :=$ (γ_2, γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6, γ_9, γ_8, γ_7). Playing $\mathbf{s}$ six times in a row results in the position $(0, 12(k+1)+3, 0, -4(k+1), -1, 0, -4(k+1), 0, 0)$. From the fundamental position $\omega_7 = (0, 0, 0, 0, 0, 0, 1, 0, 0)$, play the (legal) sequence (γ_7, γ_6, γ_5, γ_4, γ_3, γ_1, γ_8, γ_7, γ_6, γ_5, γ_4, γ_3, γ_9, γ_8, γ_7, γ_6, γ_5, γ_4) to obtain the position $(0, 3, 0, -1, 0, 0, 0, 0, 0)$. This is the $k = 0$ version of the position $(0, 3k+3, 0, -k-1, 0, 0, -k, 0, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence $\mathbf{s}$ of node firings is legal when played twice in a row: $\mathbf{s} :=$ (γ_2, γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6, γ_9, γ_8, γ_7). Playing $\mathbf{s}$ twice in a row results in the position $(0, 3(k+1)+3, 0, -(k+1)-1, 0, 0, -(k+1), 0, 0)$. From the fundamental position $\omega_8 = (0, 0, 0, 0, 0, 0, 0, 1, 0)$, play the (legal) sequence (γ_8, γ_7, γ_6, γ_5, γ_4, γ_3, γ_1, γ_9, γ_8, γ_7, γ_6, γ_5, γ_4, γ_3, γ_2, γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6, γ_9, γ_8, γ_7, γ_2 γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6) to obtain the position $(0, 4, 0, -1, 0, -1, 0, 0, 0)$. This is the $k = 0$ version of the position $(0, 6k+4, 0, -2k-1, 0, -1, -2k, 0, 0)$. From any such position with $k \geq 0$, play the game to see that the following sequence $\mathbf{s}$ of node firings is legal when played six times in a row: $\mathbf{s} :=$ (γ_2, γ_4, γ_3, γ_1, γ_5, γ_4, γ_3, γ_6, γ_5, γ_4, γ_7, γ_6, γ_5, γ_8, γ_7, γ_6, γ_9, γ_8, γ_7). Playing $\mathbf{s}$ six times in a row results in the position $(0, 6(k+1)+4, 0, -2(k+1)-1, 0, -1, -2(k+1), 0, 0)$. The fundamental position $\omega_9 = (0, 0, 0, 0, 0, 0, 0, 0, 1)$ is the $k = 0$ version of the position $(0, 0, 0, 0, 0, 0, 0, -k, 2k+1)$.

From any such position with $k \geq 0$, play the game to see that the following sequence of node firings is legal: (γ_9 γ_8, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_1, γ_4, γ_3, γ_5, γ_4, γ_2, γ_6, γ_5, γ_4, γ_3, γ_1, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7, γ_8, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_1, γ_4, γ_3, γ_5, γ_4, γ_2, γ_6, γ_5, γ_4, γ_3, γ_1, γ_7, γ_6, γ_5, γ_4, γ_2, γ_3, γ_4, γ_5, γ_6, γ_7, γ_8). This results in the position $(0,0,0,0,0,0,0,-(k+1),2(k+1)+1)$.

$\boxed{\text{The } \widetilde{\mathsf{F}} \text{ family}}$ First, we show why is not admissible. Label the nodes as γ_1, γ_2, γ_3, γ_4, and γ_5 from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1,0,0,0,0)$ is the $k = 0$ version of the position $(2k+1,-k,0,0,0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_1, γ_2, γ_3, γ_2, γ_4, γ_3, γ_2, γ_5, γ_4, γ_3, γ_2). This results in the position $(2(k+1)+1,-(k+1),0,0,0)$. The fundamental position $\omega_2 = (0,1,0,0,0)$ is the $k = 0$ version of the position $(-2k,4k+1,-2k,-2k,-2k)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_2, γ_3, γ_4, γ_5, γ_1, γ_2, γ_3, γ_4, γ_5, γ_1, γ_2, γ_3, γ_4, γ_5, γ_1). This results in the position $(-2(k+1),4(k+1)+1,-2(k+1),-2(k+1),-2(k+1))$. The fundamental position $\omega_3 = (0,0,1,0,0)$ is the $k = 0$ version of the position $(-k,-k,3k+1,-k,-k)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_3, γ_4, γ_5, γ_2, γ_1, γ_3, γ_4, γ_5, γ_2, γ_1, γ_3, γ_4, γ_5, γ_2, γ_1). This results in the position $(-(k+1),-(k+1),3(k+1)+1,-(k+1),-(k+1))$. The fundamental position $\omega_4 = (0,0,0,1,0)$ is the $k = 0$ version of the position $(-k,-k,k,2k+1,-k)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_4, γ_5, γ_3, γ_2, γ_1, γ_4, γ_5, γ_3, γ_2, γ_1, γ_4, γ_5, γ_3, γ_2, γ_1). This results in the position $(-(k+1),-(k+1),k+1,2(k+1)+1,-(k+1))$. The fundamental position $\omega_5 = (0,0,0,0,1)$ is the $k = 0$ version of the position $(0,0,0,-k,2k+1)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_5, γ_4, γ_3, γ_2, γ_3, γ_4, γ_1, γ_2, γ_3, γ_4, γ_2, γ_3, γ_1, γ_2, γ_3, γ_4). This results in the position $(0,0,0,-(k+1),2(k+1)+1)$.

To finish our analysis of the $\widetilde{\mathsf{F}}$ family, we show why is not admissible. Label the nodes as γ_1, γ_2, γ_3, γ_4, and γ_5 from left to right. For each fundamental position, we exhibit a divergent game sequence as a short sequence of legal node firings which can be repeated indefinitely. The fundamental position $\omega_1 = (1,0,0,0,0)$ is the $k = 0$ version of the position $(2k+1,-k,0,0,0)$. From any such position with $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_1, γ_2, γ_3, γ_2, γ_4, γ_3, γ_2, γ_5, γ_4, γ_3, γ_2). This results in the position $(2(k+1)+1,-(k+1),0,0,0)$. The fundamental position $\omega_2 = (0,1,0,0,0)$ is the $k = 0$ version of the position $(-2k,4k+1,-k,-k,-k)$. From any such position $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_2, γ_3, γ_4, γ_5, γ_1, γ_2, γ_3, γ_4, γ_5, γ_1, γ_2, γ_3, γ_4, γ_5, γ_1). This results in the position $(-2(k+1),4(k+1)+1,-(k+1),-(k+1),-(k+1))$. The fundamental position $\omega_3 = (0,0,1,0,0)$ is the $k = 0$ version of the position $(-2k,-2k,3k+1,-k,-k)$. From any such position $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_3, γ_4, γ_5, γ_2, γ_1, γ_3, γ_4, γ_5, γ_2, γ_1, γ_3, γ_4, γ_5, γ_2, γ_1). This results in the position $(-2(k+1),-2(k+1),3(k+1)+1,-(k+1),-(k+1))$. The fundamental position $\omega_4 = (0,0,0,1,0)$ is the $k = 0$ version of the position $(0,0,0,2k+1,-4k)$. From any such position $k \geq 0$, the following sequence of node firings is easily seen to be legal: (γ_4, γ_3, γ_2, γ_3, γ_4, γ_1, γ_2, γ_3, γ_4, γ_2, γ_3, γ_1, γ_2, γ_3, γ_4, γ_5). This results in the position $(0,0,0,2(k+1)+1,-4(k+1))$. The fundamental position $\omega_5 = (0,0,0,0,1)$ is the $k = 0$ version of the position $(0,0,0,-k,2k+1)$. From any such position $k \geq 0$, the following

sequence of node firings is easily seen to be legal: (γ_5, γ_4, γ_3, γ_2, γ_3, γ_4, γ_1, γ_2, γ_3, γ_4, γ_2, γ_3, γ_1, γ_2, γ_3, γ_4). This results in the position $(0, 0, 0, -(k + 1), 2(k + 1) + 1)$.

The $\widetilde{\mathsf{G}}$ family First, we show why is not admissible. Label the nodes as γ_1, γ_2, and γ_3 from left to right. A position (a, b, c) meets condition $(*)$ if $a \leq 0$, $b \leq 0$, and $a+2b+c > 0$. The following inequalities are immediate: (1) $c > 0$, (2) $b + c > 0$, (3) $a + 3b + 3c > 0$, (4) $a+2b+2c > 0$, (5) $2a+3b+3c > 0$, (6) $a+b+c > 0$, and (7) $2a+4b+3c > 0$. From (1) through (6) it now follows that all node firings of the sequence $\mathbf{s} := (\gamma_3, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ are legal: The left-hand side of each inequality is the number at the respective node of the sequence when that node is fired. The resulting position (a_1, b_1, c_1) has $a_1 = a$, $b_1 = -(a+b+c)$, and $c_1 = 2a+4b+3c$. Clearly $a_1 \leq 0$. By inequality (6), it follows that $b_1 < 0$. From inequality (7) we get $c_1 > 0$. Finally, $a_1 + 2b_1 + c_1 = a + 2b + c > 0$, so (a_1, b_1, c_1) meets condition $(*)$. So from any position which meets condition $(*)$, the firing sequence $\mathbf{s}$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_3 = (0, 0, 1)$ meets condition $(*)$. Now take fundamental position $\omega_2 = (0, 1, 0)$ and apply the legal firing sequence $(\gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ to get the resulting position $(0, -1, 4)$. The latter meets condition $(*)$. For the fundamental position $\omega_1 = (1, 0, 0)$, apply the legal firing sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1)$ to get the resulting position $(-1, 0, 2)$. The latter meets condition $(*)$.

Next, we show why is not admissible. Our argument is similar to the previous case. Label the nodes as γ_1, γ_2, and γ_3 from left to right. A position (a, b, c) meets condition $(*)$ if $b \leq 0$, $c \leq 0$, $a+3b > 0$, and $a+b+c > 0$. The following inequalities are easy to see: (1) $a > 0$, (2) $a+b > 0$, (3) $2a+3b > 0$, (4) $a+2b > 0$, (5) $a+3b > 0$, (6) $2a+3b+c > 0$, (7) $4a+7b+2c > 0$, (8) $2a+4b+c > 0$, and (9) $11a+18b+6c > 0$. From (1) through (8) it now follows that all node firings of the sequence $\mathbf{s} := (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_3)$ are legal: The left-hand side of each inequality is the number at the respective node of the sequence when that node is fired. The resulting position (a_1, b_1, c_1) has $a_1 = 11a+18b+c$, $b_1 = b$, and $c_1 = -(2a+4b+c)$. Clearly $b_1 \leq 0$. Inequality (8) gives $c_1 < 0$. From (9) we get $a_1 > 0$. Note that $a_1 + 3b_1 = 11a + 21b + 6c = 6(a + b + c) + 5(a + 3b) > 0$. Finally, $a_1 + b_1 + c_1 = 9a + 15b + 5c = 5(a + b + c) + 2(a + 2b) + 2(a + 3b) > 0$, so (a_1, b_1, c_1) meets condition $(*)$. So from any position which meets condition $(*)$, the firing sequence $\mathbf{s}$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0)$ meets condition $(*)$. For the fundamental position $\omega_2 = (0, 1, 0)$, apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_2)$ to get the resulting position $(6, -1, 0)$. The latter meets condition $(*)$. For the fundamental position $\omega_3 = (0, 0, 1)$, apply the legal firing sequence $(\gamma_3, \gamma_2, \gamma_3)$ to get the resulting position $(6, 0, -1)$, which meets condition $(*)$.

Next, we show why is not admissible. Our argument is entirely similar to the previous case. Label the nodes as γ_1, γ_2, and γ_3 from left to right. A position (a, b, c) meets condition $(*)$ if $b \leq 0$, $c \leq 0$, $a + 3b > 0$, and $a + b + c > 0$. The following inequalities are easy to see: (1) $a > 0$, (2) $a + b > 0$, (3) $2a + 3b > 0$, (4) $a + 2b > 0$, (5) $a + 3b > 0$, (6) $2a + 3b + c > 0$, (7) $6a + 10b + 3c > 0$, (8) $4a + 7b + 2c > 0$, (9) $6a + 11b + 3c > 0$, (10) $2a + 4b + c > 0$, and (11) $35a + 60b + 18c > 0$. From (1) through (10) it now follows that all node firings of the sequence $\mathbf{s} := (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_3, \gamma_2, \gamma_3)$ are legal: The left-hand side of each inequality is the number

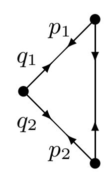
at the respective node of the sequence when that node is fired. The resulting position (a_1, b_1, c_1) has $a_1 = 35a + 60b + 18c$, $b_1 = b$, and $c_1 = -(2a + 4b + c)$. Clearly $b_1 \leq 0$. Inequality (10) gives $c_1 < 0$. From (11) we get $a_1 > 0$. Note that $a_1 + 3b_1 = 35a + 63b + 18c = 18(a+b+c) + 15(a+3b) + 2a > 0$. Finally, $a_1 + b_1 + c_1 = 33a + 57b + 17c = 17(a+b+c) + 8(a+2b) + 8(a+3b) > 0$, so (a_1, b_1, c_1) meets condition $(*)$. So from any position which meets condition $(*)$, the firing sequence $\mathbf{s}$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0)$ meets condition $(*)$. For the fundamental position $\omega_2 = (0, 1, 0)$, apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_2, \gamma_3, \gamma_2)$ to get the resulting position $(12, -1, 0)$. The latter meets condition $(*)$. For the fundamental position $\omega_3 = (0, 0, 1)$, apply the legal firing sequence $(\gamma_3, \gamma_2, \gamma_3, \gamma_2, \gamma_3)$ to get the resulting position $(18, 0, -1)$, which meets condition $(*)$.

Next, we show why is not admissible. Our argument is entirely similar to the previous case. Label the nodes as γ_1, γ_2, and γ_3 from left to right. A position (a, b, c) meets condition $(*)$ if $b \leq 0$, $c \leq 0$, $a + 3b > 0$, and $3a + 6b + c > 0$. The following inequalities are easy to see: (1) $a > 0$, (2) $a + b > 0$, (3) $2a + 3b > 0$, (4) $a + 2b > 0$, (5) $a + 3b > 0$, (6) $6a + 9b + c > 0$, (7) $6a + 10b + c > 0$, (8) $12a + 21b + 2c > 0$, (9) $6a + 11b + c > 0$, (10) $6a + 12b + c > 0$, and (11) $35a + 60b + 6c > 0$. From (1) through (10) it now follows that all node firings of the sequence $\mathbf{s} := (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_3, \gamma_2, \gamma_3)$ are legal: The left-hand side of each inequality is the number at the respective node of the sequence when that node is fired. The resulting position (a_1, b_1, c_1) has $a_1 = 35a + 60b + 6c$, $b_1 = b$, and $c_1 = -(6a + 12b + c)$. Clearly $b_1 \leq 0$. Inequality (10) gives $c_1 < 0$. From (11) we get $a_1 > 0$. Note that $a_1 + 3b_1 = 35a + 63b + 6c = 6(3a + 6b + c) + 9(a + 3b) + 8a > 0$. Finally, $3a_1 + 6b_1 + c_1 = 99a + 174b + 17c = 17(3a + 6b + c) + 24(2a + 3b) > 0$, so (a_1, b_1, c_1) meets condition $(*)$. So from any position which meets condition $(*)$, the firing sequence $\mathbf{s}$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0)$ meets condition $(*)$. For the fundamental position $\omega_2 = (0, 1, 0)$, apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_2, \gamma_3, \gamma_2)$ to get the resulting position $(12, -1, 0)$, which meets condition $(*)$. For the fundamental position $\omega_3 = (0, 0, 1)$, apply the legal firing sequence $(\gamma_3, \gamma_2, \gamma_3, \gamma_2, \gamma_3)$ to get the resulting position $(6, 0, -1)$, which meets condition $(*)$.

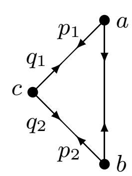
Next, we show why is not admissible. Our argument is entirely similar to the previous case. Label the nodes as γ_1, γ_2, and γ_3 from left to right. A position (a, b, c) meets condition $(*)$ if $b \leq 0$, $c \leq 0$, $a+3b > 0$, and $2a+4b+c > 0$. The following inequalities are easy to see: (1) $a > 0$, (2) $a+b > 0$, (3) $2a+3b > 0$, (4) $a+2b > 0$, (5) $a+3b > 0$, (6) $4a+6b+c > 0$, (7) $4a+7b+c > 0$, (8) $4a+8b+c > 0$, and (9) $11a+18b+3c > 0$. From (1) through (8) it now follows that all node firings of the sequence $\mathbf{s} := (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_3, \gamma_2, \gamma_3)$ are legal: The left-hand side of each inequality is the number at the respective node of the sequence when that node is fired. The resulting position (a_1, b_1, c_1) has $a_1 = 11a+18b+3c$, $b_1 = b$, and $c_1 = -(4a+8b+c)$. Clearly $b_1 \leq 0$. Inequality (8) gives $c_1 < 0$. From (9) we get $a_1 > 0$. Note that $a_1+3b_1 = 11a+21b+3c = 3(2a+4b+c)+3(a+3b)+2a > 0$. Finally, $2a_1 + 4b_1 + c_1 = 18a + 32b + 5c = 5(2a + 4b + c) + 4(2a + 3b) > 0$, so (a_1, b_1, c_1) meets condition $(*)$. So from any position which meets condition $(*)$, the firing sequence $\mathbf{s}$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal

node firings. The fundamental position $\omega_1 = (1,0,0)$ meets condition $(*)$. For the fundamental position $\omega_2 = (0,1,0)$, apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_2)$ to get the resulting position $(6,-1,0)$. The latter meets condition $(*)$. For the fundamental position $\omega_3 = (0,0,1)$, apply the legal firing sequence $(\gamma_3, \gamma_2, \gamma_3)$ to get the resulting position $(3,0,-1)$, which meets condition $(*)$.

To finish our analysis of the $\widetilde{\mathsf{G}}$ family, we show why is not admissible. Our argument is similar to the previous case. Label the nodes as γ_1, γ_2, and γ_3 from left to right. A position (a,b,c) meets condition $(*)$ if $a \leq 0$, $b \leq 0$, and $3a+2b+c > 0$. The following inequalities are easy to see: (1) $c > 0$, (2) $b+c > 0$, (3) $a+b+c > 0$, (4) $3a+2b+2c > 0$, (5) $2a+b+c > 0$, (6) $3a+b+c > 0$, and (7) $6a+4b+3c > 0$. From (1) through (6) it now follows that all node firings of the sequence $\mathbf{s} := (\gamma_3, \gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ are legal: The left-hand side of each inequality is the number at the respective node of the sequence when that node is fired. The resulting position (a_1, b_1, c_1) has $a_1 = a$, $b_1 = -(3a+b+c)$, and $c_1 = 6a+4b+3c$. Clearly $a_1 \leq 0$. Inequality (6) gives $b_1 < 0$. Note that $3a_1 + 2b_1 + c_1 = 3a+2b+c > 0$, so (a_1, b_1, c_1) meets condition $(*)$. So from any position which meets condition $(*)$, the firing sequence $\mathbf{s}$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_3 = (0,0,1)$ meets condition $(*)$. For the fundamental position $\omega_2 = (0,1,0)$, apply the legal firing sequence $(\gamma_2, \gamma_1, \gamma_2, \gamma_1, \gamma_2)$ to get the resulting position $(0,-1,4)$, which meets condition $(*)$. For the fundamental position $\omega_1 = (1,0,0)$, apply the legal firing sequence $(\gamma_1, \gamma_2, \gamma_1, \gamma_2, \gamma_1)$ to get the resulting position $(-1,0,6)$, which meets condition $(*)$.

$\boxed{\text{Families of small cycles}}$ First, we show why GCM graphs of the form 

are not admissible.

Assign numbers a, b, and c as follows: 

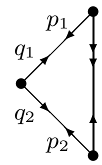
Set $\kappa := (p_1 + p_2 - \frac{1}{q_2})a + (p_1 + p_2 - \frac{1}{q_1})b + c$.

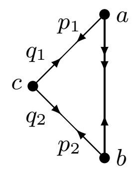
Assume for now that $a \geq 0$, $b \geq 0$, $c \leq 0$, and $\kappa > 0$; when these inequalities hold we will say the position (a,b,c) meets condition $(*)$. Under condition $(*)$ notice that a and b cannot both be zero. Begin by firing only at the two rightmost nodes. When this is no longer possible, fire at the leftmost node. The resulting corresponding numbers are $a_1 = q_1(\kappa + \frac{1}{q_2}a)$, $b_1 = q_2(\kappa + \frac{1}{q_1}b)$, and $c_1 = -\kappa - \frac{1}{q_2}a - \frac{1}{q_1}b$. In particular, $a_1 > 0$, $b_1 > 0$, and $c_1 < 0$. Next we check that $\kappa_1 := (p_1 + p_2 - \frac{1}{q_2})a_1 + (p_1 + p_2 - \frac{1}{q_1})b_1 + c_1$ is also positive. Now

$$\kappa_1 = Q\kappa + Q_1 a + Q_2 b,$$

where $Q = q_1(p_2 - \frac{1}{q_2}) + q_2(p_1 - \frac{1}{q_1}) + (p_1q_1 + p_2q_2 - 1)$, $Q_1 = \frac{1}{q_2}[q_1(p_2 - \frac{1}{q_2}) + (p_1q_1 - 1)]$, and $Q_2 = \frac{1}{q_1}[q_2(p_1 - \frac{1}{q_1}) + (p_2q_2 - 1)]$. Since each parenthesized quantity in our expression for Q is nonnegative and the last of these is positive, then $Q > 0$. Similar reasoning shows that each bracketed quantity in our expressions for Q_1 and Q_2 is nonnegative, hence $Q_1 \geq 0$ and $Q_2 \geq 0$. Since $\kappa > 0$ by hypothesis, it now follows that $\kappa_1 > 0$. Then (a_1, b_1, c_1) meets condition $(*)$, so we can legally repeat the above firing sequence from position (a_1, b_1, c_1) to obtain another position (a_2, b_2, c_2) that meets condition $(*)$, etc. Since the fundamental positions $(a,b,c) = (1,0,0)$ and

$(a,b,c) = (0,1,0)$ meet condition $(*)$, then we see that the indicated legal firing sequence can be repeated indefinitely from these positions. For the fundamental position $(a,b,c) = (0,0,1)$, begin by firing at the leftmost node to obtain the position $(q_1, q_2, -1)$. This latter position meets condition $(*)$ with $\kappa = Q$, and so the legal firing sequence indicated above can be repeated indefinitely from this position.

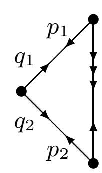
Next, we show why GCM graphs of the form 

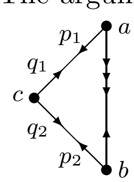
are not admissible. We assume that the amplitude products p_1q_1 and p_2q_2 are at least two. The argument is entirely similar to the previous case. Assign numbers a, b, and c as follows: 

Set $\kappa := (2p_1+2p_2-\frac{1}{q_1})a+(p_1+2p_2-\frac{1}{q_2})b+c$. Assume for now that $a \geq 0$, $b \geq 0$, $c \leq 0$, and $\kappa > 0$; when these inequalities hold we will say the position (a,b,c) meets condition $(*)$. Using the same firing sequence as before, the resulting corresponding numbers are $a_1 = q_1(\kappa+\frac{1}{q_2}b)$, $b_1 = q_2(\kappa+\frac{1}{q_1}a)$, and $c_1 = -\kappa-\frac{1}{q_1}a-\frac{1}{q_2}b$. In particular, $a_1 > 0$, $b_1 > 0$, and $c_1 < 0$. Next we check that $\kappa_1 := (2p_1 + 2p_2 - \frac{1}{q_1})a_1 + (p_1 + 2p_2 - \frac{1}{q_2})b_1 + c_1$ is also positive. Now

$$\kappa_1 = Q\kappa + Q_1a + Q_2b,$$

where $Q = q_1(2p_2 - \frac{1}{q_1}) + q_2(p_1 - \frac{1}{q_2}) + (2p_1q_1 + 2p_2q_2 - 1)$, $Q_1 = \frac{1}{q_1}[q_2(p_1 - \frac{1}{q_2}) + (2p_2q_2 - 1)]$, and $Q_2 = \frac{1}{q_2}[q_1(2p_2 - \frac{1}{q_1}) + (2p_1q_1 - 1)]$. Since each parenthesized quantity in our expression for Q is nonnegative and the last of these is positive, then $Q > 0$. Similar reasoning shows that each bracketed quantity in our expressions for Q_1 and Q_2 is nonnegative, hence $Q_1 \geq 0$ and $Q_2 \geq 0$. Since $\kappa > 0$ by hypothesis, it now follows that $\kappa_1 > 0$. Conclude as in the previous case.

Next, we show why GCM graphs of the form 

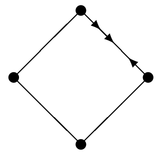
are not admissible. We assume that the amplitude products p_1q_1 and p_2q_2 are at least three. The argument is entirely similar to the previous two cases. Assign numbers a, b, and c as follows: 

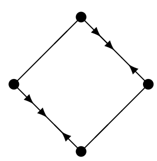
Set $\kappa := (4p_1 + 6p_2 - \frac{1}{q_1})a + (2p_1 + 4p_2 - \frac{1}{q_2})b + c$. Assume for now that $a \geq 0$, $b \geq 0$, $c \leq 0$, and $\kappa > 0$; when these inequalities hold we will say the position (a,b,c) meets condition $(*)$. Using the same firing sequence as in the previous two cases, the resulting corresponding numbers are $a_1 = q_1(\kappa + \frac{1}{q_2}b)$, $b_1 = q_2(\kappa + \frac{1}{q_1}a)$, and $c_1 = -\kappa - \frac{1}{q_1}a - \frac{1}{q_2}b$. In particular, $a_1 > 0$, $b_1 > 0$, and $c_1 < 0$. Next we check that $\kappa_1 := (4p_1 + 6p_2 - \frac{1}{q_1})a_1 + (2p_1 + 4p_2 - \frac{1}{q_2})b_1 + c_1$ is also positive. Now

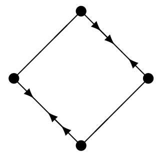
$$\kappa_1 = Q\kappa + Q_1a + Q_2b,$$

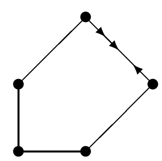
where $Q = q_1(6p_2 - \frac{1}{q_1}) + q_2(2p_1 - \frac{1}{q_2}) + (4p_1q_1 + 4p_2q_2 - 1)$, $Q_1 = \frac{1}{q_1}[q_2(2p_1 - \frac{1}{q_2}) + (4p_2q_2 - 1)]$, and $Q_2 = \frac{1}{q_2}[q_1(6p_2 - \frac{1}{q_1}) + (4p_1q_1 - 1)]$. Since each parenthesized quantity in our expression for

Q is nonnegative and the last of these is positive, then $Q > 0$. Similar reasoning shows that each bracketed quantity in our expressions for Q_1 and Q_2 is nonnegative, hence $Q_1 \geq 0$ and $Q_2 \geq 0$. Since $\kappa > 0$ by hypothesis, it now follows that $\kappa_1 > 0$. Conclude as in the previous two cases.

Next, we show why  is not admissible. The argument is similar to the previous three cases, but simpler since the amplitudes are all known. Number the nodes with γ_1 as the North vertex, γ_2 as the East vertex, γ_3 as the South vertex, and γ_4 as the West vertex. We say an initial position (a, b, c, d) meets condition $(*)$ if the following inequalities are satisfied: $b \geq 0$, $c \geq 0$, $d \leq 0$, and $a + d > 0$. The firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is easily seen to be legal from any such position. The resulting position is (a_1, b_1, c_1, d_1) with $a_1 = 4a + 2b + c + d$, $b_1 = c$, $c_1 = a + d$, and $d_1 = -(3a + b + c + d)$. It is easy now to check that (a_1, b_1, c_1, d_1) also meets condition $(*)$. (In fact, the inequalities $c_1 > 0$ and $d_1 < 0$ are now strict.) So from any position which meets condition $(*)$, the firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0, 0)$ meets condition $(*)$. Now take fundamental position $\omega_2 = (0, 1, 0, 0)$ and apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(2, 0, 0, -1)$. The latter meets condition $(*)$. Next take fundamental position $\omega_3 = (0, 0, 1, 0)$ and apply the legal firing sequence $(\gamma_3, \gamma_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(5, 0, 0, -3)$. The latter meets condition $(*)$. For fundamental position $\omega_4 = (0, 0, 0, 1)$, apply the legal firing sequence $(\gamma_4, \gamma_1, \gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(4, 1, 0, -3)$. The latter meets condition $(*)$.

Next, we show why  is not admissible. The argument is entirely similar to the previous case. Number the nodes with γ_1 as the North vertex, γ_2 as the East vertex, γ_3 as the South vertex, and γ_4 as the West vertex. We say an initial position (a, b, c, d) meets condition $(*)$ if the following inequalities are satisfied: $a > 0$, $b \geq 0$, $c \geq 0$, $d \leq 0$, and $3a+b+c+2d > 0$. The firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is easily seen to be legal from any such position. The resulting position is (a_1, b_1, c_1, d_1) with $a_1 = 4a + 2b + c + d$, $b_1 = c$, $c_1 = 4a + b + c + 2d$, and $d_1 = -(3a + b + c + d)$. It is easy now to check that (a_1, b_1, c_1, d_1) also meets condition $(*)$. (In fact, the inequalities $c_1 > 0$ and $d_1 < 0$ are now strict.) So from any position which meets condition $(*)$, the firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0, 0)$ meets condition $(*)$. Now take fundamental position $\omega_2 = (0, 1, 0, 0)$ and apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(2, 0, 1, -1)$. The latter meets condition $(*)$. Next take fundamental position $\omega_3 = (0, 0, 1, 0)$ and apply the legal firing sequence (γ_3, γ_4) to get the resulting position $(1, 1, 1, -1)$. The latter meets condition $(*)$. For fundamental position $\omega_4 = (0, 0, 0, 1)$, apply the legal firing sequence (γ_4) to get the resulting position $(1, 0, 2, -1)$. The latter meets condition $(*)$.

Next, we show why  is not admissible. The argument is entirely similar to the previous case. Number the nodes with γ_1 as the North vertex, γ_2 as the East vertex, γ_3 as the South vertex, and γ_4 as the West vertex. We say an initial position (a, b, c, d) meets condition $(*)$ if the following inequalities are satisfied: $a > 0$, $b \geq 0$, $c \geq 0$, $d \leq 0$, and $3a+b+c+d > 0$. The firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ is easily seen to be legal from any such position. The resulting position is (a_1, b_1, c_1, d_1) with $a_1 = 6a+3b+2c+d$, $b_1 = c$, $c_1 = 3a+b+c+d$, and $d_1 = -(5a+2b+2c+d)$. It is easy now to check that (a_1, b_1, c_1, d_1) also meets condition $(*)$. (In fact, the inequalities $c_1 > 0$ and $d_1 < 0$ are now strict.) So from any position which meets condition $(*)$, the firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0, 0)$ meets condition $(*)$. Now take fundamental position $\omega_2 = (0, 1, 0, 0)$ and apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_4)$ to get the resulting position $(3, 0, 0, -2)$. The latter meets condition $(*)$. Next take fundamental position $\omega_3 = (0, 0, 1, 0)$ and apply the legal firing sequence (γ_3, γ_4) to get the resulting position $(2, 1, 1, -2)$. The latter meets condition $(*)$. For fundamental position $\omega_4 = (0, 0, 0, 1)$, apply the legal firing sequence (γ_4) to get the resulting position $(1, 0, 1, -1)$. The latter meets condition $(*)$.

To finish our analysis of families of small cycles, we show why  is not admissible. The argument is entirely similar to the previous case. Number the nodes with γ_1 as the North vertex and γ_2, γ_3, γ_4, and γ_5 in succession in the clockwise order around the cycle. We say an initial position (a, b, c, d, e) meets condition $(*)$ if the following inequalities are satisfied: $b \geq 0$, $c \geq 0$, $d \geq 0$, $e \leq 0$, and $a + e > 0$. The firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ is easily seen to be legal from any such position. The resulting position is $(a_1, b_1, c_1, d_1, e_1)$ with $a_1 = 4a + 2b + c + d + e$, $b_1 = c$, $c_1 = d$, $d_1 = a + e$, and $e_1 = -(3a + b + c + d + e)$. It is easy now to check that $(a_1, b_1, c_1, d_1, e_1)$ also meets condition $(*)$. (In fact, the inequalities $d_1 > 0$ and $e_1 < 0$ are now strict.) So from any position which meets condition $(*)$, the firing sequence $(\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ can be legally applied indefinitely, resulting in a divergent game sequence. Then it suffices to show that from each fundamental position we can reach a position which meets condition $(*)$ using a sequence of legal node firings. The fundamental position $\omega_1 = (1, 0, 0, 0, 0)$ meets condition $(*)$. Now take fundamental position $\omega_2 = (0, 1, 0, 0, 0)$ and apply the legal firing sequence $(\gamma_2, \gamma_3, \gamma_4, \gamma_5)$ to get the resulting position $(2, 0, 0, 0, -1)$. The latter meets condition $(*)$. Next take fundamental position $\omega_3 = (0, 0, 1, 0, 0)$ and apply the legal firing sequence $(\gamma_3, \gamma_4, \gamma_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ to get the resulting position $(5, 0, 0, 0, -3)$. The latter meets condition $(*)$. Next take fundamental position $\omega_4 = (0, 0, 0, 1, 0)$ and apply the legal firing sequence $(\gamma_4, \gamma_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ to get the resulting position $(4, 1, 0, 0, -3)$. The latter meets condition $(*)$. For fundamental position $\omega_4 = (0, 0, 0, 0, 1)$, apply the legal firing sequence $(\gamma_5, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5)$ to get the resulting position $(4, 0, 1, 0, -3)$. The latter meets condition $(*)$.

This completes the proof of Proposition 3.1. $\square$

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